

# On the $k$ -gamma $q$ -distribution

Rafael Díaz, Camilo Ortiz, and Eddy Pariguan

## Abstract

We provide combinatorial as well as probabilistic interpretations for the  $q$ -analogue of the Pochhammer  $k$ -symbol introduced by Díaz and Teruel. We introduce  $q$ -analogues of the Mellin transform in order to study the  $q$ -analogue of the  $k$ -gamma distribution.

## 1 Introduction

There is a general strategy for building bridges between combinatorics and measure theory which we describe. Let  $d\mu$  be a measure on a interval  $I \subseteq [0, \infty) \subseteq \mathbb{R}$ . We say that  $d\mu$  is a combinatorial measure if for each  $n \in \mathbb{N}$  the  $n$ -th moment of  $d\mu$  is a non-negative integer. Equivalently, let  $M_{d\mu}$  be the Mellin transform of  $d\mu$  given by

$$M_{d\mu}(t) = \int_I x^{t-1} d\mu.$$

Then  $d\mu$  is a combinatorial measure if and only if  $M_{d\mu}(n) \in \mathbb{N}$  for  $n \in \mathbb{N}_+$ .

Let  $cmeas$  be the set of combinatorial measures, and consider the map  $m : cmeas \rightarrow \mathbb{N}^{\mathbb{N}}$  that sends a combinatorial measure into its moment's sequence  $(m_0, \dots, m_n, \dots)$ . Recall [14] that a sequence of finite sets  $(s_0, \dots, s_n, \dots)$  provides a combinatorial interpretation for a sequence of integers  $(m_0, \dots, m_n, \dots)$  if it is such that  $|s_n| = m_n$ . By analogy we say that the sequence of finite sets  $(s_0, \dots, s_n, \dots)$  provides a combinatorial interpretation for a measure  $d\mu$  if for each  $n \in \mathbb{N}$  the following identity holds:

$$|s_n| = \int_I x^n d\mu.$$

In this work we consider the reciprocal problem: given  $m = (m_0, \dots, m_n, \dots) \in \mathbb{N}^{\mathbb{N}}$  find a combinatorial interpretation for it, and furthermore find a combinatorial measure  $d\mu$  such that its sequence of moments is  $m$ . Our main goal is to establish an instance of the correspondence combinatorics/measure theory described above within the context of  $q$ -calculus. Namely, we are going to study the combinatorial and the measure theoretic interpretations for the  $k$ -increasing factorial  $q$ -numbers  $[1]_{n,k} = [1]_q [1+k]_q [1+2k]_q \cdots [1+(n-1)k]_q \in \mathbb{N}[q]^{\mathbb{N}}$  which are obtained as an instance of the  $q$ -analogue of the Pochhammer  $k$ -symbol given by

$$[t]_{n,k} = [t]_q [t+k]_q [t+2k]_q \cdots [t+(n-1)k]_q = \prod_{j=0}^{n-1} [t+jk]_q$$

where  $[t]_q = \frac{1-q^t}{1-q}$  is the  $q$ -analogue of  $t$ . The search for the combinatorial and measure theoretic interpretation for the  $k$ -increasing factorial  $q$ -numbers must be made within the context of  $q$ -calculus; this means that we have to broaden our techniques in order to include  $q$ -combinatorial interpretations, and the  $q$ -analogues for the Lebesgue's measure and the Mellin's transform.

## 2 The $k$ -gamma measure

Perhaps the best known example of the relation combinatorics/measure theory discussed in the introduction comes from the factorial numbers  $n!$  which count, respectively, the number of elements of  $S_n$ , the group of permutations of a set with  $n$  elements. The Mellin transform of the measure  $e^{-x}dx$  is the classical gamma function given for  $t > 0$  by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$

The moments of the measure  $e^{-x}dx$  are precisely the factorial numbers, indeed we have that

$$|S_n| = n! = \Gamma(n+1) = \int_0^\infty x^n e^{-x} dx.$$

Notice that  $n! = (1)_n$ , where the Pochhammer symbol  $(t)_n$  is given by

$$(t)_n = t(t+1)(t+2) \dots (t+(n-1)).$$

As a second example [5] consider the combinatorial and measure theoretical interpretations for the  $k$ -increasing factorial numbers  $(1)_{n,k} = (1+k)(1+2k) \dots (1+(n-1)k)$ , which arise as an instance of the Pochhammer  $k$ -symbol given by

$$(t)_{n,k} = t(t+k)(t+2k) \dots (t+(n-1)k) = \prod_{j=0}^{n-1} (t+jk).$$

The combinatorics of the Pochhammer  $k$ -symbol has attracted considerable attention in the literature, from the work of Gessel and Stanley [10] up to the quite recent works [2, 12]. Assume  $t$  is a non-negative integer and let  $T_{n,k}^t$  be the set of isomorphism classes of planar rooted trees  $T$  such that: 1) The set of internal vertices, i.e. vertices with one outgoing edge and at least one incoming edge, of  $T$  is  $\{1, 2, \dots, n\}$ ; 2)  $T$  has a unique vertex with no outgoing edges called the root;  $T$  has a set  $L(T)$  of vertices called leaves, the leaves have no incoming edges; 3) The valence of each internal vertex of  $T$  is  $k+2$ ; 4) The valence of the root is  $t$ ; 5) If the internal vertex  $i$  is on the path from the internal vertex  $j$  to the root, then  $i < j$ .

Note that the set of leaves  $L(T)$  comes with a natural order, and thus we can assign a number between 1 to  $|L(T)|$  to each leaf. Figure 1 shows an example of a graph in  $T_{4,2}^2$ .

One can show by induction that  $(t)_{n,k} = |T_{n,k}^t|$ .

The Mellin transform of the measure  $e^{-\frac{x}{k}}dx$  is the  $k$ -gamma function  $\Gamma_k$  given for  $t > 0$  by

$$\Gamma_k(t) = \int_0^\infty x^{t-1} e^{-\frac{x}{k}} dx.$$

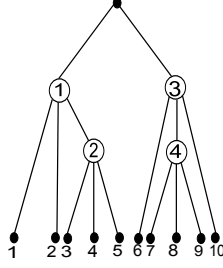


Figure 1: Example of a tree in  $T_{4,2}^2$ .

The  $k$ -gamma function  $\Gamma_k : (0, \infty) \rightarrow \mathbb{R}$  is univocally determined [5] by the following properties:  $\Gamma_k(t+k) = t\Gamma_k(t)$  for  $t \in \mathbb{R}^+$ ;  $\Gamma_k(k) = 1$ ;  $\Gamma_k$  is logarithmically convex. See [11, 13] for further properties of the  $k$ -gamma function.

The  $k$ -increasing factorial numbers appear as moments of the  $\Gamma_k$  function as follows:

$$|T_{n,k}^1| = (1)_{n,k} = \Gamma_k(1+nk) = \frac{1}{\Gamma_k(1)} \int_0^\infty x^{nk} e^{\frac{-x^k}{k}} dx = \frac{k^{\frac{k-1}{k}}}{\Gamma(\frac{1}{k})} \int_0^\infty x^{nk} e^{\frac{-x^k}{k}} dx.$$

Indeed the following more general identity holds:

$$|T_{n,k}^t| = (t)_{n,k} = \frac{\Gamma_k(t+nk)}{\Gamma_k(t)} = \frac{1}{\Gamma_k(t)} \int_0^\infty x^{t+nk-1} e^{\frac{-x^k}{k}} dx.$$

### 3 Review of $q$ -calculus

In this section we introduce some useful basic definitions [1, 3, 9]. We begin introducing the  $q$ -derivative and the Jackson  $q$ -integral. Let  $\text{Map}(\mathbb{R}, \mathbb{R})$  be the real vector space of functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Fix a real number  $0 \leq q < 1$ , the  $q$ -derivative is the linear operator

$$\partial_q : \text{Map}(\mathbb{R}, \mathbb{R}) \rightarrow \text{Map}(\mathbb{R} \setminus \{0\}, \mathbb{R}) \quad \text{given by}$$

$$\partial_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \quad \text{For example we have that } \partial_0 f(x) = \frac{f(x) - f(0)}{x}.$$

Notice that  $\partial_q f$  is not a priori well-defined at  $x = 0$ . Nevertheless, it is often the case that  $\partial_q f$  can be extended by continuity over the whole real line, e.g. when  $f$  is a polynomial function.

For  $0 \leq a < b \leq +\infty$  the Jackson  $q$ -integral from  $a$  to  $b$  of  $f \in \text{Map}(\mathbb{R}, \mathbb{R})$  is given by

$$\int_a^b f(x) d_q x = (1-q)b \sum_{n=0}^{\infty} q^n f(q^n b) - (1-q)a \sum_{n=0}^{\infty} q^n f(q^n a).$$

For example we have that

$$\int_a^b f(x) d_0 x = bf(b) - af(a).$$

Set  $I_q f(x) = f(qx)$ . The following properties hold for  $f, g \in \text{Map}(\mathbb{R}, \mathbb{R})$  :

$$\begin{aligned}\partial_q(fg) &= \partial_q f g + I_q f \partial_q g \\ \partial_q(f(ax^b)) &= a[b]_q x^{b-1} \partial_q f(ax^b) \\ f(b)g(b) - f(a)g(a) &= \int_a^b \partial_q f g d_q x + \int_a^b I_q g \partial_q f d_q x,\end{aligned}$$

For  $0 < q < 1$ ,  $x, y \in \mathbb{R}$ ,  $n \in \mathbb{N}_+$ , and  $t \in \mathbb{R}$  we set

$$(x+y)_{q^k}^n = \prod_{j=0}^{n-1} (x + q^{jk}y), \quad (x+y)_{q^k}^\infty = \prod_{j=0}^{\infty} (x + q^{jk}y) \quad \text{and} \quad (1+x)_{q^k}^t = \frac{(1+x)_{q^k}^\infty}{(1+q^{kt}x)_{q^k}^\infty}.$$

## 4 $q$ -Analogue of the $k$ -gamma function

We proceed to study the  $q$ -analogue of the  $k$ -increasing factorial numbers

$$[1]_{n,k} = [1]_q [1+k]_q [1+2k]_q \cdots [1+(n-1)k]_q$$

which are an instance of the  $q$ -analogue of the Pochhammer  $k$ -symbol  $[t]_{n,k}$  given for  $t \in \mathbb{R}$  by

$$[t]_{n,k} = [t]_q [t+k]_q [t+2k]_q \cdots [t+(n-1)k]_q = \prod_{j=0}^{n-1} [t+jk]_q.$$

The motivation behind our definition of the  $q$ -analogue of the  $k$ -gamma function comes from the work of De Sole and Kac [4], where they introduced a  $q$ -deformation of the gamma function given by the  $q$ -integral:

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q^{-qx} d_q x,$$

where the  $q$ -analogue  $E_q^x$  of the exponential function is given by

$$E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}.$$

For example we have that  $E_0^x = 1+x$ ,  $E_0^{-0x} = 1$ , and therefore  $\Gamma_0(t) = 1$ .

We define the  $q$ -analogue of the  $k$ -gamma function  $\Gamma_{q,k}$  by demanding that it satisfies the  $q$ -analogues of the properties of the  $\Gamma_k$  function. Thus  $\Gamma_{q,k}$  is such that  $\Gamma_{q,k}(t+k) = [t]_q \Gamma_{q,k}(t)$  and  $\Gamma_{q,k}(k) = 1$ . Several applications of the former property show that

$$\Gamma_{q,k}(nk) = \prod_{j=1}^{n-1} [jk]_q = \prod_{j=1}^{n-1} \frac{(1-q^{jk})}{(1-q)} = \frac{(1-q^k)_{q^k}^{n-1}}{(1-q)^{n-1}}.$$

After a change of variables the function  $\Gamma_{q,k}$  may be written as follows:

$$\Gamma_{q,k}(t) = \frac{(1-q^k)_{q^k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}}. \quad \text{For example we have that } \Gamma_{0,k}(t) = 1.$$

The previous formula implies an infinite product expression for  $\Gamma_{q,k}$  given by

$$\Gamma_{q,k}(t) = \frac{(1-q)^{1-\frac{t}{k}}(1-q^k)_{q^k}^{\infty}}{(1-q^t)_{q^k}^{\infty}},$$

and also the following result.

**Lemma 1.** The  $q, k$ -gamma function  $\Gamma_{q,k}$  and the  $q^k$ -gamma function  $\Gamma_{q^k}$  are related by the identity  $\Gamma_{q,k}(t) = [k]_q^{\frac{t}{k}-1} \Gamma_{q^k}(\frac{t}{k})$ .

The following result [8] provides an integral representation for  $\Gamma_{q,k}$ .

**Proposition 2.**

$$\Gamma_{q,k}(t) = \int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} x^{t-1} E_{q^k}^{-\frac{q^k x^k}{[k]_q}} d_q x.$$

This integral representation for  $\Gamma_{q,k}$  may be regarded as a  $q$ -analogue of the Mellin transform, therefore one is entitled to consider the  $q$ -measure

$$E_{q^k}^{-\frac{q^k x^k}{[k]_q}} d_q x \quad \text{defined on the interval} \quad \left[0, \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}\right]$$

as the inverse Mellin  $q$ -transform of the  $\Gamma_{q,k}$  function. Figure 4 shows the graph of  $E_{q^k}^{-\frac{q^k x^k}{[k]_q}}$  for  $q = 0.6$  and  $1 \leq k \leq 5$ .

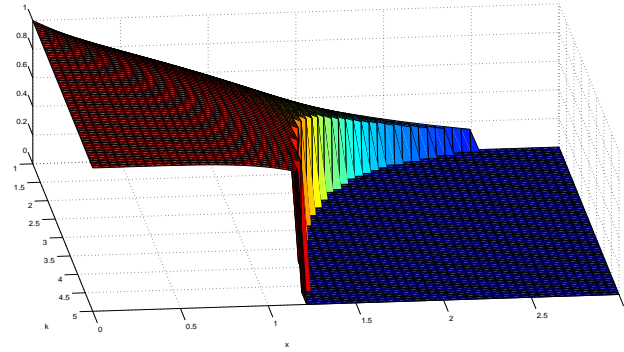


Figure 2: Display of  $E_{q^k}^{-\frac{q^k x^k}{[k]_q}}$  for  $q = 0.6$  and  $1 \leq k \leq 5$ .

One can check that for  $0 \leq x \leq \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}$  the function  $E_{q^k}^{-\frac{q^k x^k}{[k]_q}}$  is given by

$$E_{q^k}^{-\frac{q^k x^k}{[k]_q}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} x^{kn}}{[k]_q^n [n]_{q^k}!}.$$

**Theorem 3.** The function  $\Gamma_{q,k}$  is given by

$$\Gamma_{q,k}(t) = (1-q)^{1-\frac{t}{k}} \sum_{n=0}^{\infty} \frac{q^{\frac{kn(n+1)}{2}}}{(1-q^{kn+t})(q^k-1)^n [n]_{q^k}!}.$$

*Proof.* From Theorem 7 below we know that

$$\int_0^x s^{t-1} E_{q^k}^{-\frac{q^k s^k}{[k]_q}} d_q s = (1-q)x^t \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} x^{kn}}{(1-q^{kn+t})[k]_q^n [n]_{q^k}!}.$$

The desired result follows taking  $x = \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}$ . □

**Corollary 4.**

$$(1-q^k)^{\frac{t}{k}}_{q^k} = \sum_{n=0}^{\infty} \frac{q^{\frac{kn(n+1)}{2}}}{(1-q^{kn+t})(q^k-1)^n [n]_{q^k}!}.$$

*Proof.* Follows from Theorem 3 and the identity  $\Gamma_{q,k}(t) = \frac{(1-q^k)^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}}$ . □

By definition the cumulative distribution function associated with the measure

$$E_{q^k}^{-\frac{q^k x^k}{[k]_q}} d_q x \quad \text{is given for} \quad 0 \leq x \leq \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}} \quad \text{by} \quad \int_0^x E_{q^k}^{-\frac{q^k s^k}{[k]_q}} d_q s.$$

**Proposition 5.**

$$\int_0^x E_{q^k}^{-\frac{q^k s^k}{[k]_q}} d_q s = (1-q)x \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} x^{kn+1}}{(1-q^{kn+1})[k]_q^n [n]_{q^k}!}.$$

*Proof.* The result follows from Theorem 7 below taking  $t = 1$ . □

## 5 Combinatorial interpretation of the Pochhammer $q, k$ -symbol

Just as in combinatorics one studies the cardinality of finite sets, in  $q$ -combinatorics one studies the cardinality of  $q$ -weighted finite sets, i.e. pairs  $(x, \omega)$  where  $x$  is a finite set and the  $q$ -weight is an arbitrary map  $\omega : x \rightarrow \mathbb{N}[q]$  from  $x$  to  $\mathbb{N}[q]$  the algebra of polynomials in  $q$  with non-negative integer coefficients. The cardinality of the pair  $(x, \omega)$  is by definition given by

$$|x, \omega| = \sum_{i \in x} \omega(i) \in \mathbb{N}[q].$$

To provide a  $q$ -combinatorial interpretation for the Pochhammer  $k$ -symbol  $[t]_{n,k}$  we let again  $t$  be a positive integer and consider the set  $T_{n,k}^t$  of planar rooted trees introduced above. Next we define a  $q$ -weight  $\omega$  on  $T_{n,k}^t$ . The construction of  $\omega$  is based on the following elementary facts:

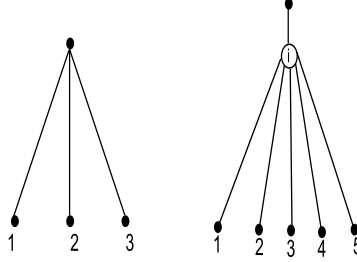


Figure 3: Display of the tree  $r_3$  and a tree  $c_i$  with 5 leaves.

- 1) Let  $r_t$  be the rooted tree with  $t$  leaves and no internal vertices. See Figure 3.
- 2) For  $1 \leq i \leq n$ , let  $c_i$  be the rooted tree with  $i$  as its unique internal vertex and  $k + 1$  leaves. See Figure 3.
- 3) If  $T$  is a planar rooted tree and  $l$  is a number between 1 and  $|L(T)|$ , then there is a well-defined rooted planar tree  $T \circ_l c_i$  obtained by gluing the root of  $c_i$  with the leaf  $l$  of  $T$  to form a new edge.
- 4) Clearly each tree  $T \in \mathbb{T}_{n,k}^t$  can be written in a unique way as

$$T = (\dots((r_t \circ_{l_1} c_1) \circ_{l_2} c_2) \dots) \circ_{l_n} c_n.$$

- 5) The weight  $\omega(T)$  of a tree  $T$  written in the form above is given by

$$\omega(T) = \prod_{i=1}^{n-1} q^{l_i-1} \in \mathbb{N}[q].$$

For the tree  $T$  from Figure 1 we have that

$$T = (((r_2 \circ_1 c_1) \circ_3 c_2) \circ_6 c_3) \circ_7 c_4 \quad \text{and} \quad \omega(T) = q^0 q^2 q^5 q^6 = q^{13}.$$

**Theorem 6.**

$$[t]_{n,k} = |\mathbb{T}_{n,k}^t, \omega|.$$

*Proof.* The proof goes by induction on  $n$ . We have the following chain of identities

$$|\mathbb{T}_{n+1,k}^t, \omega| = \sum_{T \in \mathbb{T}_{n+1,k}^t} \omega(T) = \sum_{S \in \mathbb{T}_{n,k}^t} \sum_{l \in L(S)} \omega(S \circ_l c_{n+1}) = \sum_{S \in \mathbb{T}_{n,k}^t} \sum_{l=1}^{t+kn} \omega(S) q^{l-1} =$$

$$= \left( \sum_{S \in \mathbb{T}_{n,k}^t} \omega(S) \right) \left( \sum_{l=1}^{t+nk} q^{l-1} \right) = |\mathbb{T}_{n+1,k}^t, \omega| [t+nk]_q = [t]_{n,k} [t+nk]_q = [t]_{n+1,k}.$$

In the computation above we used two main facts: 1) Each tree  $\mathbb{T}_{n,k}^t$  has exactly  $t+nk$  leaves; 2) Each tree  $T \in \mathbb{T}_{n+1,k}^t$  can be written in a unique way as  $T = S \circ_l c_{n+1}$  where  $S \in \mathbb{T}_{n,k}^t$ ,  $l$  is a leaf of  $S$ , and  $c_{n+1}$  is the rooted tree with  $k+1$  leaves and  $n+1$  as its unique internal vertex.  $\square$

## 6 $k$ -Gamma $q$ -distribution

We are ready to define the  $k$ -gamma  $q$ -distribution. From the identity

$$\Gamma_{q,k}(t) = \int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} x^{t-1} E_{q^k}^{-\frac{q^k x^k}{[k]_q}} d_q x$$

we see that the function

$$x^{t-1} \frac{E_{q^k}^{-\frac{q^k x^k}{[k]_q}}}{\Gamma_{q,k}(t)}$$

defines a  $q$ -density on the interval  $[0, \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}]$ , in the sense that it is a non-negative function whose  $q$ -integral is equal to one. Consider the case  $t = 1$  and  $k = 3$ . Figure 6 shows the graph of  $\frac{E_{q^3}^{-\frac{q^3 x^3}{[3]_q}}}{\Gamma_{q,3}(1)}$  for  $q \in [0, 1)$ .

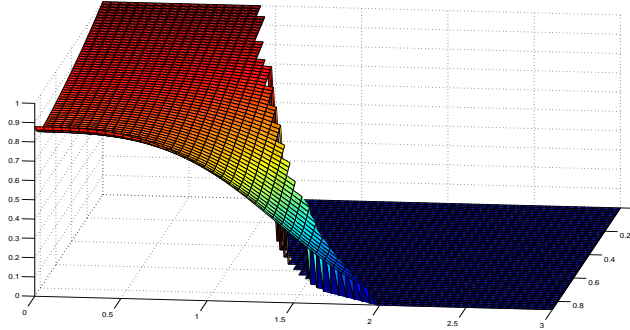


Figure 4: Display of  $\frac{E_{q^3}^{-\frac{q^3 x^3}{[3]_q}}}{\Gamma_{q,3}(1)}$  for  $q \in [0, 1)$ .

**Theorem 7.** The cumulative distribution of the  $k$ -gamma  $q$ -density is given by

$$\frac{1}{\Gamma_{q,k}(t)} \int_0^x s^{t-1} E_{q^k}^{-\frac{q^k s^k}{[k]_q}} d_q s = \frac{(1-q)x^t}{\Gamma_{q,k}(t)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} x^{kn}}{[k]_q^n [n]_{q^k}! (1-q^{kn+t})}.$$



*Proof.*

$$\begin{aligned}
\frac{1}{\Gamma_{q,k}(t)} \int_0^x s^{t-1} E_{q^k}^{-\frac{q^k s^k}{[k]_q}} d_q s &= \frac{(1-q)x}{\Gamma_{q,k}(t)} \sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} (q^m x)^{kn+t-1}}{[k]_q^n [n]_{q^k}!} \\
&= \frac{1-q}{\Gamma_{q,k}(t)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} x^{kn+t}}{[k]_q^n [n]_{q^k}!} \sum_{m=0}^{\infty} q^{m(kn+t)} \\
&= \frac{1-q}{\Gamma_{q,k}(t)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} x^{kn+t}}{[k]_q^n [n]_{q^k}! (1 - q^{kn+t})}
\end{aligned}$$

□

Consider the case  $t = 1$  and  $k = 3$ . Figure 6 shows the cumulative distribution associated to the  $q$ -density  $\frac{E_{q^3}^{-\frac{q^3 x^3}{[3]_q}}}{\Gamma_{q,3}(1)}$  for  $q \in [0, 1)$ .

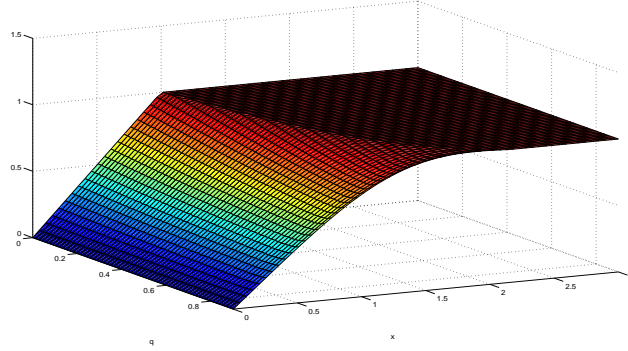


Figure 5: Cumulative distribution of the  $q$ -density  $\frac{E_{q^3}^{-\frac{q^3 x^3}{[3]_q}}}{\Gamma_{q,3}(1)}$  for  $q \in [0, 1)$ .

The previous considerations imply our next result which establishes an example of the link between  $q$ -combinatorics and  $q$ -measure theory promised in the introduction.

**Theorem 8.** The  $k$ -increasing factorial  $q$ -numbers appear as moments of the  $\Gamma_{q,k}$  function as follows:

$$|\mathbb{T}_{n,k}^1, \omega| = [1]_{n,k} = \Gamma_{q,k}(1 + nk) = \frac{1}{\Gamma_{q,k}(1)} \int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} x^{nk} E_{q^k}^{-\frac{q^k x^k}{[k]_q}} d_q x.$$

Indeed the following more general identity also holds:

$$|\mathbb{T}_{n,k}^t, \omega| = [t]_{n,k} = \frac{\Gamma_{q,k}(t + nk)}{\Gamma_{q,k}(t)} = \frac{1}{\Gamma_{q,k}(t)} \int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} x^{t+nk-1} E_{q^k}^{-\frac{q^k x^k}{[k]_q}} d_q x.$$

## 7 $k$ -Beta $q$ -distribution

Recall that the classical beta function is given for  $s, t > 0$  by

$$B(t, s) = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)} = \int_0^1 x^{t-1}(1-x)^{s-1} dx.$$

The  $q$ -analogue of the  $k$ -beta function is correspondingly defined by

$$B_{q,k}(t, s) = \frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{\Gamma_{q,k}(t+s)} = \frac{(1-q)(1-q^k)_{q^k}^{\frac{s}{k}-1}}{(1-qt)_{q^k}^{\frac{s}{k}}}.$$

Notice that  $B_{0,k}(t, s) = 1$ . One can show [8] that the function  $B_{q,k}$  has the following integral representation

$$B_{q,k}(t, s) = [k]_q^{-\frac{t}{k}} \int_0^{[k]_q^{\frac{1}{k}}} x^{t-1} \left(1 - q^k \frac{x^k}{[k]_q}\right)_{q^k}^{\frac{s}{k}-1} d_q x.$$

Because of the factor  $[k]_q^{-\frac{t}{k}}$  this integral representation is not quite a Mellin transform. However we see that the  $q$ -measure

$$\left(1 - q^k \frac{x^k}{[k]_q}\right)_{q^k}^{\frac{s}{k}-1} d_q x$$

is a Mellin  $q$ -transformation inverse of the function

$$B_{q,k}(t, s)[k]_q^{\frac{t}{k}}.$$

On the other hand we see that the function

$$\frac{x^{t-1} \left(1 - q^k \frac{x^k}{[k]_q}\right)_{q^k}^{\frac{s}{k}-1}}{B_{q,k}(t, s)[k]_q^{\frac{t}{k}}}$$

defines a  $q$ -density on the interval  $[0, [k]_q^{\frac{1}{k}}]$ , indeed it defines a  $q$ -analogue for the  $k$ -beta density. Figure 7 below shows the graph of the  $q$ -density

$$\frac{x^{-0.5} \left(1 - q^3 \frac{x^3}{[3]_q}\right)_{q^3}^{\frac{0.5}{3}-1}}{B_{q,3}(0.5, 0.5)[3]_q^{\frac{0.5}{3}}}$$

Our final result provides an explicit formula for the cumulative beta  $q$ -distribution. It follows as an easy consequence of the definition of the Jackson integral and the definition of  $B_{q,k}(t, s)$ .

**Theorem 9.** The cumulative beta  $q$ -distribution is given by

$$\frac{1}{B_{q,k}(t, s)[k]_q^{\frac{t}{k}}} \int_0^x s^{t-1} \left(1 - q^k \frac{s^k}{[k]_q}\right)_{q^k}^{\frac{s}{k}-1} d_q x = \frac{(1-q)x^t}{B_{q,k}(t, s)[k]_q^{\frac{t}{k}}} \sum_{n=0}^{\infty} q^{nt} \left(1 - q^{k(n+1)} \frac{x^k}{[k]_q}\right)_{q^k}^{\frac{s}{k}-1}.$$

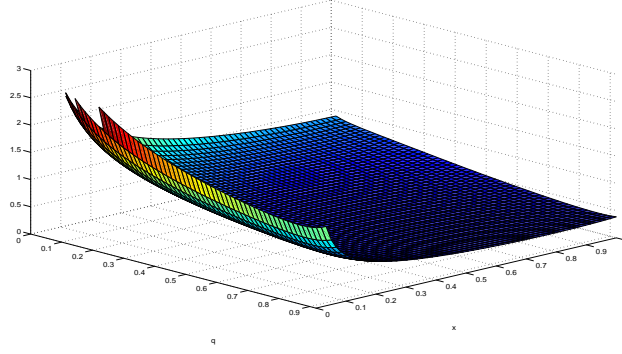


Figure 6: Display of the function  $\frac{x^{-0.5} \left(1 - q^3 \frac{x^3}{[3]_q}\right)^{\frac{0.5}{3} - 1} q^3}{B_{q,3}(0.5, 0.5) [3]_q^{\frac{0.5}{3}}}$  for  $0 \leq q < 1$  and  $0 \leq x \leq 1$ .

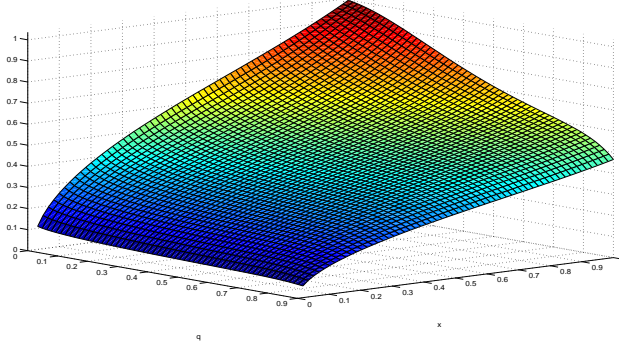


Figure 7: Cumulative distribution of the  $q$ -density  $\frac{x^{-0.5} \left(1 - q^3 \frac{x^3}{[3]_q}\right)^{\frac{0.5}{3} - 1} q^3}{B_{q,3}(0.5, 0.5) [3]_q^{\frac{0.5}{3}}}$  for  $0 \leq q < 1$  and  $0 \leq x \leq 1$ .

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ragadiaz@gmail.com

Instituto de Matemáticas y sus Aplicaciones, Universidad Sergio Arboleda, Bogotá, Colombia

camiloortiz@javeriana.edu.co, epariguan@javeriana.edu.co

Departamento de Matemáticas, Pontificia Universidad Javeriana, Bogotá, Colombia